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Note

Extremal functions of forbidden double permutation matrices

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ABSTRACT

We say a 0–1 matrix A avoids a matrix P if no submatrix of A can be transformed into P by changing some ones to zeroes. We call P an *m-tuple permutation matrix* if P can be obtained by replacing each column of a permutation matrix with m copies of that column. In this paper, we investigate $n \times n$ matrices that avoid P and the maximum number $ex(n, P)$ of ones that they can have. We prove a linear bound on $ex(n, P)$ for any 2-tuple permutation matrix P, resolving a conjecture of Keszegh [B. Keszegh, On linear forbidden matrices, J. Combin. Theory Ser. A 116 (1) (2009) 232–241]. Using this result, we obtain a linear bound on $ex(n, P)$ for any m-tuple permutation matrix P. Additionally, we demonstrate the existence of infinitely many minimal non-linear patterns, resolving another conjecture of Keszegh from the same paper.

1. Introduction

In this paper, we examine pattern avoidance in 0–1 matrices, an outgrowth of research on permutation patterns. A 0–1 *matrix* is a matrix with all entries either 0 or 1. A *pattern* P is just a 0–1 matrix. We say that A *represents* P if A can be transformed into P by changing some ones to zeroes. A *contains* P if some submatrix of A represents P. If A does not contain P, then we say that A *avoids* P. Finally, we define the *extremal function* of P, denoted $ex(n, P)$, to be the maximum number of ones in an $n \times n$ matrix A that avoids P. In this paper, we will represent a pattern with a dot for each one and a space for each zero (see Fig. 1). It will be assumed that in each pattern, the leftmost and rightmost columns and the bottom and top rows are not empty.

Observe that for any pattern P with multiple ones, there is an $n \times n$ matrix with at least n ones that avoids P. Take an $n \times n$ matrix A with a single row of n ones and all other entries 0 and an $n \times n$

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$$\begin{matrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

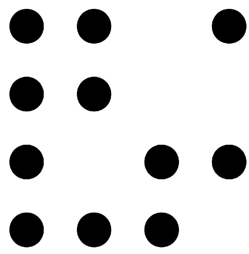

Fig. 1. A 0–1 matrix represented with zeroes and ones on the left, and with dots and spaces on the right.

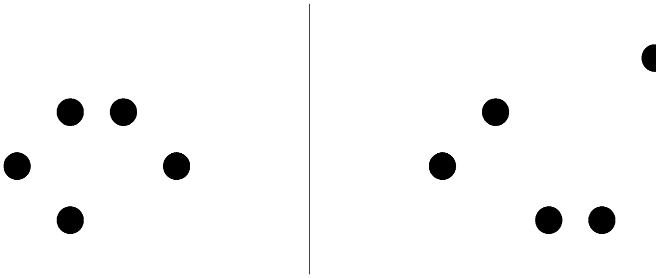

Fig. 2. Patterns L_1 and L_2, respectively.

matrix B with a single column of n ones and all other entries 0. If one of these matrices contains P, then the other one avoids P. So, besides trivial patterns with a single one, all patterns have extremal functions that grow at least linearly in n.

We use the notation $ex(n, P) = \Theta(f(n))$ if there are positive constants c and c' such that $ex(n, P)$ is bounded asymptotically from below by $cf(n)$ and from above by $c'f(n)$. Bienstock and Györi [1] examined the extremal functions of certain trapezoidal patterns with four ones, determining that for these patterns, $ex(n, P) = \Theta(n\alpha(n))$, where $\alpha(n)$ denotes the incredibly slow growing inverse Ackermann function. Füredi and Hajnal [3] asked which patterns have extremal functions that are linear in n, i.e., for what P does $ex(n, P) = \Theta(n)$? In their paper, they determined the asymptotic behavior of the extremal functions of all but a few of the patterns with at most four ones. Tardos [7], in addition to dealing with extremal functions of sets of patterns, determined the asymptotic behavior of the extremal functions of all remaining patterns with at most four ones. The combined results of these three papers show that for patterns P with at most four ones, $ex(n, P)$ ranges from 0 (a pattern with a single one) to $\Theta(n^{\frac{3}{2}})$ (a pattern with four ones in a rectangle). Additionally, both [3] and [7] exhibited some operations performed on patterns that do not significantly change the asymptotic behavior of their extremal functions. Tardos also raised the question of whether the pattern L_1, shown in Fig. 2, has a non-linear extremal function. This question was answered in the negative by Fulek [2], who also introduced a method of bounding the extremal function using planar graphs.

If a pattern P has a non-linear extremal function but all patterns properly contained by that pattern have linear or zero extremal functions, then P is called a *minimal non-linear pattern*. Keszegh [5], in an effort to determine whether certain patterns contained minimal non-linear patterns, exhibited some decompositions that can be performed on patterns that can simplify the calculation of their extremal functions. He also exhibited the first pattern with more than four ones known to be minimal non-linear. Additionally, he suggested a method for demonstrating the existence of infinitely many minimal non-linear patterns. The success of his method depends on the validity of conjectures about the linearity of extremal functions of certain patterns, including L_2 (Fig. 2). The results of this

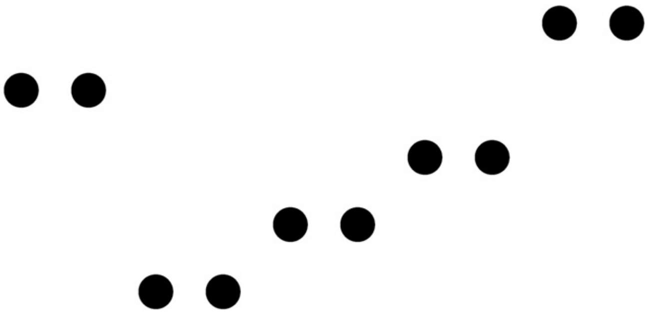

Fig. 3. A 5×10 double permutation matrix.

paper verify these conjectures and demonstrate the existence of infinitely many minimal non-linear patterns.

A permutation matrix has a single one in each column and each row. We will call P an *m-tuple permutation matrix* if P can be obtained by replacing each column of a permutation matrix with m copies of that column. In the new matrix, the copies of column i will appear in the columns between $(i-1)m+1$ and im inclusive. A 1-tuple permutation matrix is just a permutation matrix. Marcus and Tardos [6] proved that all permutation matrices have extremal functions with linear bounds. Keszegh [5] conjectured that all 2-tuple (or double) permutation matrices have linear extremal functions. By modifying the recursive method of Marcus and Tardos, we obtain a linear upper bound for all double permutation matrices, proving Keszegh's conjecture to be true. As a corollary, this also proves generally that $ex(n,P) = \Theta(n)$ for any m-tuple permutation matrix P. Furthermore, this proves the linearity of the extremal functions that Keszegh considered, and hence demonstrates the existence of infinitely many minimal non-linear patterns.

2. Main theorem

In this section, we prove that any double permutation matrix has a linear extremal function, resolving Conjecture 4.3 of [5]. The corollaries of this proof will appear in the following sections.

Let the pattern P be any $k \times 2k$ double permutation matrix with $k > 1$ (see Fig. 3 for an example).

Fix an integer n that is divisible by $2k^2$. Choose any $n \times n$ matrix A with $ex(n,P)$ ones that avoids P. We now partition A into $(\frac{n}{2k^2})^2$ submatrices, each with $2k^2$ rows and $2k^2$ columns. Define *block* S_{ij}, a square submatrix of A, to be the intersection of rows $2k^2(i-1)+1$ through $2k^2 i$ of A with columns $2k^2(j-1)+1$ through $2k^2 j$ of A.

We call S_{ij} *wide* if it has at least $2k$ ones in a single row. We call S_{ij} *tall* if it has at least k rows with ones.

Blocks that are neither wide nor tall have less than k rows with ones and less than $2k$ ones in any of these rows. Hence they have at most $(k-1)(2k-1)$ ones total.

Finally we construct an $\frac{n}{2k^2} \times \frac{n}{2k^2}$ 0–1 matrix Q. Define the ones of Q row by row from top to bottom, in each row from left to right such that $q_{ij} = 1$ if (1) S_{ij} is the leftmost block in its row of blocks with a one or (2) S_{ij} has a one and some row of A contained in the blocks between $S_{ij'}$ and S_{ij} inclusive (where j' is the greatest number less than j such that $q_{ij'} = 1$) has at least two ones. See Fig. 4 for a specific example of A and Q.

Lemma 1. *Q avoids P.*

Proof. Suppose Q contains P. For each row h of P, take the pair of ones q_{ij} and $q_{ij'}$ in row i of Q that represent the pair of ones in row h of P. Consider blocks S_{ij} and $S_{ij'}$ in the same block-row of A. By the definition of Q, some row r_h of A contained in the blocks between S_{ij} and $S_{ij'}$ inclusive has at least two ones. For each h, we select row r_h of A and two columns of A that have ones in this row

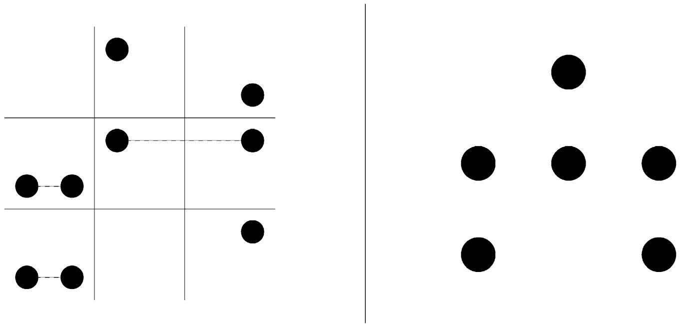

Fig. 4. A 6×6 matrix A with $k = 1$ and corresponding matrix Q.

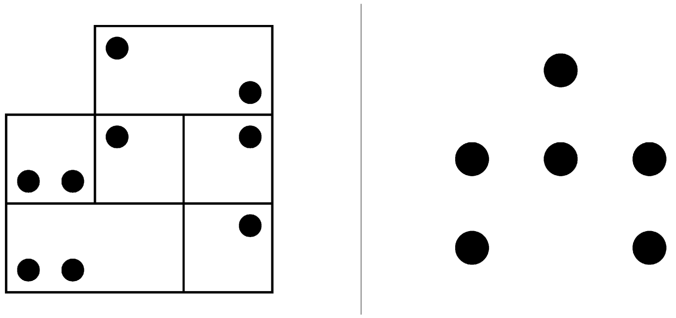

Fig. 5. A 6×6 matrix A with $k = 1$ with its non-empty blocks partitioned into chunks, and corresponding matrix Q.

contained in the blocks between S_{ij} and $S_{ij'}$ inclusive. The intersection of all selected columns with all selected rows yields a submatrix of A that represents P. This is a contradiction. $\square$

Lemma 2. *In each column of blocks $C_j = \bigcup S_{ij}$ for $1 \leqslant i \leqslant \frac{n}{2k^2}$, the number of wide blocks is less than $k\binom{2k^2}{2k}$.*

Proof. Suppose the number of wide blocks in C_j is at least $k\binom{2k^2}{2k}$. By the pigeonhole principle, there exist k blocks in C_j that have $2k$ ones in the same columns $c_1 < c_2 < \cdots < c_{2k}$ and in a single row. This $k \times 2k$ grid of ones represents P. This is a contradiction. $\square$

Lemma 3. *In each row of blocks $R_i = \bigcup S_{ij}$ for $1 \leqslant j \leqslant \frac{n}{2k^2}$, the number of tall blocks is less than $2k\binom{2k^2}{k}$.*

Proof. Suppose the number of tall blocks in R_i is at least $2k\binom{2k^2}{k}$. By the pigeonhole principle, there exist $2k$ blocks in R_i that have a one in the same rows $r_1 < r_2 < \cdots < r_k$. Call these blocks $S_{id_1}, \ldots, S_{id_{2k}}$ with $1 \leqslant d_1 < d_2 < \cdots < d_{2k} \leqslant \frac{n}{2k^2}$. For each one p_{xy} in P, take the one in row r_x of S_{id_y}. These ones represent P in A. This is a contradiction. $\square$

We introduce chunks of blocks, C_{ij}, which are sets of blocks in the same block-row R_i. A *chunk* C_{ij} consists of a block S_{ij} with $q_{ij} = 1$ on the left and all blocks, including empty ones, between it and $S_{ij'}$, the next block to the right of S_{ij} with $q_{ij'} = 1$. The chunk C_{ij} does not include $S_{ij'}$. If S_{ij} is the rightmost block in its block-row with $q_{ij} = 1$, then its chunk contains it and all blocks in its block-row to the right of it. Hence the chunks partition the non-empty blocks of A. See Fig. 5 for an example of a matrix with its non-empty blocks partitioned into chunks.

Note that if a chunk has two ones in the same row, then both ones must be in the same block. Note also that if block S_{ij} has two ones in the same row, then $q_{ij} = 1$ and $q_{ij'} = 1$ for the next non-empty $S_{ij'}$ because this row with two ones is contained between S_{ij} and $S_{ij'}$ inclusive. Hence if a chunk has two ones in the same row, then there is a single non-empty block in the chunk. So all chunks fall into two categories: (1) chunks with a single non-empty block or (2) chunks with at least two non-empty blocks and no rows with more than a single one.

Lemma 4. *In each row of blocks $R_i = \bigcup S_{ij}$ for $1 \leqslant j \leqslant \frac{n}{2k^2}$, the number of chunks of the second kind with at least k ones is less than $2k\binom{2k^2}{k}$.*

Proof. Suppose the number of chunks of the second kind with at least k ones in R_i is at least $2k\binom{2k^2}{k}$. By the pigeonhole principle, there exist $2k$ chunks in R_i that have a one in the same rows $r_1 < r_2 < \cdots < r_k$. Call these chunks $C_{id_1}, \ldots, C_{id_{2k}}$ with $1 \leqslant d_1 < d_2 < \cdots < d_{2k} \leqslant \frac{n}{2k^2}$. For each one p_{xy} in P, take the one in row r_x of C_{id_y}. These ones represent P in A. This is a contradiction. $\square$

Lemma 5. *For n divisible by $2k^2$, $ex(n, P) \leqslant \binom{2k^2}{2k}(10k^3)n + (k-1)(2k-1)ex(\frac{n}{2k^2}, P)$.*

Proof. By Lemma 2, there are at most $\frac{n}{2k^2}k\binom{2k^2}{2k} = \frac{n}{2k}\binom{2k^2}{2k}$ wide blocks, which means there are at most $\binom{2k^2}{2k}2k^3n$ ones contained in wide blocks because a single block has at most $4k^4$ ones.

By Lemma 3, there are at most $\frac{n}{2k^2}2k\binom{2k^2}{k} = \frac{n}{k}\binom{2k^2}{k}$ tall blocks, which means there are at most $\binom{2k^2}{k}4k^3n$ ones contained in tall blocks because a single block has at most $4k^4$ ones.

Every non-empty, non-tall, and non-wide block is contained in either a chunk of the first kind or the second kind. If it is contained in a chunk of the first kind, then it is the only non-empty block in that chunk, so that chunk contributes at most $(k-1)(2k-1)$ ones. By Lemma 4, at most $\frac{n}{2k^2}2k\binom{2k^2}{k} = \frac{n}{k}\binom{2k^2}{k}$ chunks of the second kind contain k or more ones (and they contain less than $4k^4$ ones), while the remaining chunks of the second kind must contain less than k ones.

Hence the number of ones from blocks that are non-empty, non-tall, and non-wide is at most $(k-1)(2k-1)ex(\frac{n}{2k^2}, P) + \binom{2k^2}{k}4k^3n$ because there are at most $ex(\frac{n}{2k^2}, P)$ chunks by Lemma 1 and $k < (k-1)(2k-1)$. So the total number of ones in A is at most

$$\binom{2k^2}{2k}2k^3n + \binom{2k^2}{k}4k^3n + (k-1)(2k-1)ex\left(\frac{n}{2k^2}, P\right) + \binom{2k^2}{k}4k^3n,$$

which is less than

$$\binom{2k^2}{2k}\left(2k^3 + 4k^3 + 4k^3\right)n + (k-1)(2k-1)ex\left(\frac{n}{2k^2}, P\right) \quad \text{because } 2 \leqslant k,$$

which implies

$$ex(n, P) \leqslant \binom{2k^2}{2k}\left(10k^3\right)n + (k-1)(2k-1)ex\left(\frac{n}{2k^2}, P\right). \qquad \square$$

Theorem 6. *$ex(n, P) \leqslant 10k^4\binom{2k^2}{2k}n$.*

Proof. We proceed by induction on n. The inequality holds trivially for $n \leqslant 2k^2$. Now assume the inequality holds for all $n < m$ and take the case $n = m$. Let N be the largest multiple of $2k^2$ less than or equal to n. By Lemma 5 and the fact that $(q(2k^2) + r)^2 - (q(2k^2))^2 = r(4k^2q + r) \leqslant 4k^2(q(2k^2) + r)$ for $0 \leqslant r < 2k^2$ and $0 \leqslant q$,

$$\begin{aligned} ex(n, P) &\leqslant ex(N, P) + 4k^2n \\ &\leqslant \binom{2k^2}{2k}\left(10k^3\right)N + (k-1)(2k-1)ex\left(\frac{N}{2k^2}, P\right) + 4k^2n \end{aligned}$$

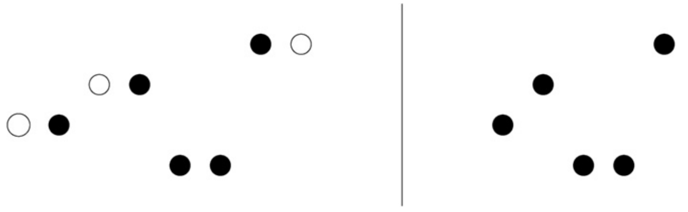

Fig. 6. A 4×8 double permutation matrix contains L_2.

$$\begin{aligned}
&\leqslant \binom{2k^2}{2k}(10k^3)N + (k-1)(2k-1)10k^4\binom{2k^2}{2k}\frac{N}{2k^2} + 4k^2 n \\
&\leqslant k^2\bigl(5(k-1)(2k-1) + 10k + 4\bigr)\binom{2k^2}{2k} n \\
&\leqslant 10k^4\binom{2k^2}{2k} n. \qquad \square
\end{aligned}$$

3. Corollaries

For our corollaries of Theorem 6, we collect some results on operations and decompositions that can be performed on patterns without significantly changing their extremal functions. Note that reflections and rotations obviously do not change extremal functions, so we can express the following lemma in various ways.

Lemma 7.

(1) *If P' contains P, then $ex(n, P) \leqslant ex(n, P')$.*
(2) *If P' is obtained from P by attaching a new row to the bottom of P and placing a single one in the new row under an existing one in the last row of P, then $ex(n, P) \leqslant ex(n, P') \leqslant ex(n, P) + n$.*
(3) *If P' is obtained from P by inserting a column between two existing columns of P and placing a single one in that column so that the new one has an existing one of P on both sides, then $ex(n, P) \leqslant ex(n, P') \leqslant 2ex(n, P)$.*
(4) *Let P and Q be two patterns such that P has a one in its lower right corner and Q has a one in its upper left corner. Let R be the pattern consisting of P in its upper left part and Q in its lower right part with exactly one common entry between them, which is the lower right one of P and the upper left one of Q. All entries in the upper right and lower left of R are blank. Then $\max(ex(n, P), ex(n, Q)) \leqslant ex(n, R) \leqslant ex(n, P) + ex(n, Q)$.*
(5) *We say that P' **reduces** to P if we can obtain P from P' by removing all empty rows and columns. If P' is finite and reduces to P, then $ex(n, P) \leqslant ex(n, P') = O(ex(n, P) + n)$.*

Proof. The proof of Lemma 7(1) is quite short: if a matrix A avoids P, then it also avoids P'. Lemma 7(2) is proven in [3, 2.2] by Füredi and Hajnal. Lemmas 7(3) and 7(5) are proven in [7, 2.3] by Tardos. The final lemma, Lemma 7(4), is proven in [4, 3.1] and [5, 2.2] by Keszegh. $\square$

We now move on to the actual corollaries.

Corollary 8. $ex(n, L_2) = \Theta(n)$.

Proof. Take the double permutation matrix for $k = 4$ shown in Fig. 6. L_2 is represented by a submatrix of this pattern, as shown in the figure, and hence has extremal function at most the extremal function of the pattern by Lemma 7(1), which is $\Theta(n)$ by Theorem 6. This resolves Conjecture 4.1.1 of [5]. $\square$

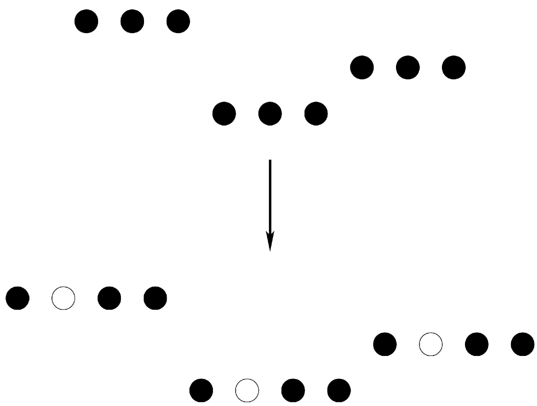

Fig. 7. Applying Lemma 7(3) in the inductive step between a triple permutation matrix and a quadruple permutation matrix.

Corollary 9. *For a $k \times mk$ m-tuple permutation matrix P, $ex(n, P) = \Theta(n)$.*

Proof. Marcus and Tardos [6] proved the result for $m = 1$ and Theorem 6 proves the result for $m = 2$. For the remaining cases, we proceed by induction with base case $m = 2$: Suppose for all $k \times mk$ m-tuple permutation matrices P, the extremal function is bounded from above by a linear function. If P' is a $k \times (m+1)k$ $(m+1)$-tuple permutation matrix, let P be the m-tuple permutation matrix obtained by deleting a single one from every row (and the column that contains it). Applying Lemma 7(3), we insert a column with a single one between two ones in each row of P (Fig. 7) to re-obtain P', changing the extremal function by at most a factor of 2^k, hence preserving its linearity. $\square$

4. The existence of infinitely many minimal non-linear patterns

In order to demonstrate the existence of infinitely many minimal non-linear patterns, we must define an infinite family of patterns, $H_k = (h_{ij})$ for $k \geqslant 0$ (Fig. 8), introduced by Keszegh [5]. The matrix H_k has $3k+4$ rows and $3k+4$ columns and is symmetrical to the line going from its top right corner to lower left corner. All entries are zeroes except for:

$$h_{41} = h_{12} = h_{13} = h_{(3k+3)(3k+4)} = h_{(3k+2)(3k+4)} = 1,$$

$$h_{(3i+4)(3i+1)} = h_{(3i-1)(3i+3)} = h_{(3i)(3i+2)} = 1 \quad \text{for } 1 \leqslant i \leqslant k.$$

This yields a total of $3k+5$ ones in H_k.

Keszegh [5] demonstrates that $ex(n, H_k) = \Theta(n \log n)$. Rather than showing that H_k is a minimal non-linear pattern, we will show that H_k contains a large enough minimal non-linear pattern to get our desired result. We will slightly modify Keszegh's method for showing the existence of infinitely many minimal non-quasilinear patterns in [5] to prove our result.

Lemma 10. *For $k \geqslant 1$, there are at least $k+5$ ones in H_k such that removing any one of them gives a pattern with a linear extremal function.*

Proof. If we remove the one at $h_{(3k+4)(3k+1)}$ from H_k, this new matrix reduces to a matrix that can be obtained by adding a row with a single one in column $3k+4$ to the bottom of the reduced matrix consisting of the ones of H_k except for $h_{(3k+4)(3k+1)}$ and $h_{(3k+3)(3k+4)}$. But this other reduced matrix with two ones removed is just a permutation matrix with a single doubled column (columns 2 and 3), and hence it is contained by a double permutation matrix. Thus removing the one at $h_{(3k+4)(3k+1)}$ results in a pattern with linear extremal function by Theorem 6, Lemmas 7(1), 7(2), and 7(5). By symmetry, removing the one at h_{41} also results in a pattern with linear extremal function.

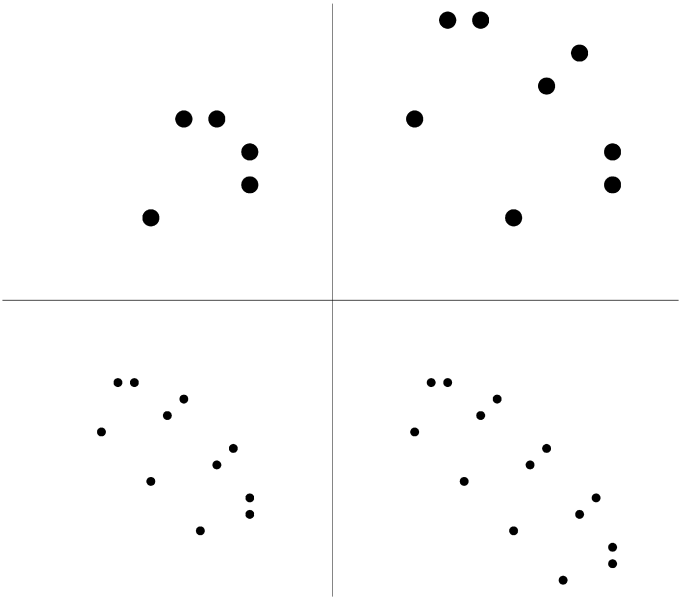

Fig. 8. Patterns H_0, H_1, H_2, and H_3.

If we remove the one at $h_{(3k+2)(3k+4)}$ or $h_{(3k+3)(3k+4)}$, we are left with a matrix that reduces to a permutation matrix with a single doubled column (columns 2 and 3). This pattern is contained by a double permutation matrix, and hence has linear extremal function by Theorem 6 and Lemmas 7(1) and 7(5). By symmetry, removing the one at h_{12} or h_{13} also results in a pattern with linear extremal function.

Now we remove the one at $h_{(3i+4)(3i+1)}$ for $1 \leqslant i \leqslant k-1$ from H_k and reduce the matrix, obtaining new pattern H_k^i. This pattern can be decomposed into two patterns P' and Q'. The pattern P' is the intersection of the first $(3i+1)$ rows and first $(3i+2)$ columns of H_k^i, while Q' is the intersection of the final $(3(k-i)+2)$ rows and final $(3(k-i)+1)$ columns of H_k^i. It should be noted that P' has ones at

$$h_{41},\ h_{12},\ h_{13},\ \text{and}\ h_{(3j+4)(3j+1)},\ h_{(3j-1)(3j+3)},\ h_{(3j)(3j+2)}\ \text{for}\ 1 \leqslant j \leqslant i\ \text{except for}\ h_{(3i+4)(3i+1)},$$

while Q' has the ones at

$$h_{(3k+2)(3k+4)},\ h_{(3k+3)(3k+4)},\ \text{and}\ h_{(3j+4)(3j+1)},\ h_{(3j-1)(3j+3)},\ h_{(3j)(3j+2)}\ \text{for}\ i < j \leqslant k.$$

Let P be obtained from P' by adding a single row after the last row of P' and a single column after the last column of P' with a single one in their intersection. Similarly, let Q be obtained from Q' by adding a single row before the first row of Q' and a single column before the first column of Q' with a single one in their intersection. Observe that P is a permutation matrix with one column doubled, and Q is the rotation of a permutation matrix with one column doubled. Hence P is contained in a double permutation matrix and Q is contained in the rotation of a double permutation matrix, so by Theorem 6 and Lemma 7(1) they both have linear extremal functions. The pattern R obtained by applying Lemma 7(4) to P and Q also has a linear extremal function. Finally, H_k^i is contained by R, so H_k^i has a linear extremal function by Lemma 7(1). Since H_k^i was the pattern obtained by removing the one at $h_{(3i+4)(3i+1)}$ from H_k and then reducing that matrix, then we have that removing the one at $h_{(3i+4)(3i+1)}$ for $1 \leqslant i \leqslant k-1$ from H_k yields a pattern with linear extremal function by Lemma 7(5).

We have demonstrated that there are $2+4+(k-1)=(k+5)$ ones such that removing any one of them gives a pattern with a linear extremal function. $\square$

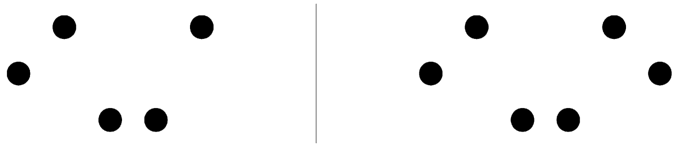

Fig. 9. Patterns L_3 and L_4, respectively.

Corollary 11. *There are infinitely many minimal non-linear patterns.*

Proof. Starting with H_k, we remove ones (and empty rows and columns) until we obtain a minimal non-linear pattern $H_k^{'}$. By Lemma 10, we can obtain a pattern with at least $k+5$ and at most $3k+5$ ones (the number of ones in H_k). We can construct an infinite sequence $1 \leqslant k_1 < k_2 < \cdots$ such that $[k_i+5, 3k_i+5] \cap [k_j+5, 3k_j+5] = \phi$ for $i \neq j$. For example, let $k_i = 4^i$. Therefore $H'_{k_i} \neq H'_{k_j}$ for $i \neq j$, and the result follows. This resolves Conjecture 4.2.1 of [5]. □

One can obtain pattern G_k from H_k by deleting the bottom three rows of H_k for $k > 1$. Then G_k is a permutation matrix with a single doubled column (columns 2 and 3) and thus is contained in a double permutation matrix, so $ex(n, G_k) = \Theta(n)$ by Theorem 6 and Lemma 7(1). This resolves Conjecture 4.1.2 of [5]. It should be noted that G_1 is just a rotation of L_2.

5. Conclusions and open problems

Corollary 9 and Lemma 7(1) give us a linear upper bound for the extremal functions of any patterns contained by an m-tuple permutation matrix, in particular any matrices with a single one in each column such that if two ones in the same row have a column between them then that column has a one in the same row.

Füredi and Hajnal's question in [3] about which patterns P have extremal functions $ex(n, P)$ which are linear in n still remains open, as do many more specific questions. For example, Fulek [2] introduced two patterns, L_3 and L_4 (Fig. 9), noting that if one could prove the linearity of $ex(n, L_3)$, then the linearity of $ex(n, L_2)$ would follow from Lemmas 7(1) and 7(2). Though we now know that $ex(n, L_2)$ is linear, this does not necessarily give us the same result for L_3. However, it should be noted that if one can prove $ex(n, L_4)$ is linear, then $ex(n, L_3)$ is linear by Lemma 7(1).

Additionally, Keszegh [5] conjectures that each H_k is a minimal non-linear pattern. Though Corollary 11 shows the existence of infinitely many minimal non-linear patterns contained in the H_k's, it would be interesting to see if the H_k's themselves are indeed minimal non-linear.

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